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LYAPUNOV'S FIRST METHOD FOR STRONGLY NON-LINEAR SYSTEMS†

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Lyapunov's classical First Method is developed for strongly non-linear systems. Techniques for "truncating" strongly non-linear systems that possess a well-defined group of symmetries are described. Given such a group, it is possible, under fairly general assumptions, to determine, by purely algebraic methods, particular solutions of the truncated systems with prescribed asymptotic expansions. It is shown that these solutions can be extended to solutions of the full system by using certain series. Sufficient conditions for the existence of parametric families of solutions of the full system that possess certain asymptotic properties are also derived. The theory is illustrated by a wide range of examples. A new proof is given of one of the inversions of the Lagrange–Dirichlet theorem on the stability of equilibrium. It is shown that the method developed here may also be used to construct collision trajectories in problems of celestial mechanics in real time.

1. INTRODUCTION

We shall consider a certain dynamical system, described by a vector differential equation with infinitely a differentiable right-hand side, for which the origin $\mathbf{x} = \mathbf{0}$ is a position of equilibrium

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}(\mathbf{0}) = \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^n \tag{1.1}$$

Using Lyapunov's First Method [1], one can constructively find families of solutions of system (1.1) that "touch" the position of equilibrium $\mathbf{x} = \mathbf{0}$. The behaviour pattern of the corresponding trajectories implies several qualitative conclusions regarding the structure of the phase portrait of the system in the neighbourhood of the equilibrium position. In particular, if there are trajectories "leaving" the position of equilibrium, the latter is unstable.

Let $\mathbf{A} = d\mathbf{f}(\mathbf{0})$ be the Jacobian of the vector field $\mathbf{f}(\mathbf{x})$ evaluated at the position of equilibrium $\mathbf{x} = \mathbf{0}$. If the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{E})$ has p roots $\lambda_1, \ldots, \lambda_p$ with negative (positive) real part, system (1.1) has a p-parameter family of particular solutions that tend to the origin as $t \to +\infty$ ($t \to -\infty$). These solutions may be expressed as series that converge for $t > 0$ ($t < 0$)

$$\mathbf{x}(t) = \sum_{j_1,\ldots,j_p=0}^{+\infty} \mathbf{x}_{j_1,\ldots,j_p}(t)\exp((j_1\lambda_1 + \ldots + j_p\lambda_p)t) \tag{1.2}$$

where $j_1 + \ldots + j_p \geqslant 1$, the coefficients $\mathbf{x}_{j_1,\ldots,j_p}(t)$ depend on t as polynomials and contain p arbitrary parameters whose numerical values must be small enough if the series (1.2) is to converge. Under these conditions the first partial sum of (1.2) is a linear combination of particular solutions of the "truncated" linear system

$$\mathbf{x}^{\cdot} = \mathbf{A}\mathbf{x} \tag{1.3}$$

the other coefficients $\mathbf{x}_{j_1,\ldots,j_p}(t)$ being determined recurrently.

It should be noted that formula (1.2) generally represents complex solutions of system (1.1). If the goal is to find real solutions, the appropriate series will look much more complicated.

Lyapunov's First Method in the classical, "quasi-linear" setting actually comprises three main steps:

1. single out a truncated (linear) system (1.3);
2. find some family of particular solutions of the truncated system;
3. complete the solutions of the truncated system found in the previous step to solutions of the full system by expressing the latter as series.

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In the theory of dynamical systems, a system is usually said to be strongly non-linear if the behaviour of one of its fixed points is not determined solely by the properties of a linearization of the corresponding vector field in the neighbourhood of the fixed point. For example, according to this terminology, systems that have what are known as asymptotic trajectories, i.e. trajectories that tend to a position of equilibrium as $t \to +\infty$ ($t \to -\infty$), but not exponentially, must be considered under the heading of strongly non-linear systems. The next problem is to find sufficient conditions for the existence and constructive determination of such trajectories.

Below we shall consider only the "supercritical" case, in which the matrix **A** of the linear approximating system has only zero roots. At first glance this might seem to be a very special situation in the great multitude of known critical cases. However, it can be shown that any critical case can be reduced to this "hyper-degenerate" situation.

Indeed, let us decompose the phase space $\mathbb{R}^n$ of the system as a direct sum of invariant subspaces of **A**

$$\mathbb{R}^n = E^{(s)} \oplus E^{(u)} \oplus E^{(c)}$$

(s stands for "stable", u for "unstable," and c for "centre"), so that the spectrum of the restriction $\mathbf{A}^{(s)} = \mathbf{A}_{E^{(s)}}$ lies in the open left half-plane, the spectrum of $\mathbf{A}^{(c)} = \mathbf{A}_{E^{(u)}}$ in the right open half-plane and the spectrum of $\mathbf{A}^{(c)} = \mathbf{A}_{E^{(c)}}$ on the imaginary axis.

If the subspace $E^{(c)}$ is of non-zero dimension, system (1.1) has a non-trivial invariant manifold, called a central manifold, tangent to $E^{(c)}$, such that the matrix of the linear part of the vector field representing the reduction of dynamical system of the central manifold is $\mathbf{A}^{(c)}$ [2]. Therefore, in order to seek non-exponential asymptotic solutions, one need only consider the system on the central manifold, and it may be assumed without loss of generality that all the roots of the linear approximating system (1.3) are purely imaginary.

The next step is to reduce the system to Poincaré normal form [3]. Let us transform system (1.1) by applying the transformation $\mathbf{x} \to \mathbf{y}$, expressed as formal power series, preserving the linear part. The matrix **A** of the linear approximation may be expressed as $\mathbf{A} = \mathbf{D} + \mathbf{J}$, where **D** is the diagonalizable part and **J** is similar to a Jordan matrix with zeros along the diagonal. System (1.1) now has the appearance

$$\mathbf{y}^{\cdot} = \mathbf{D}\mathbf{y} + \mathbf{g}(\mathbf{y}) \tag{1.4}$$

where $\mathbf{g}(\mathbf{y}) = \mathbf{J}\mathbf{y} + \dots$, the dots representing all the non-linear terms.

Definition 1. We shall say that system (1.4) has Poincaré normal form if the following formal identity holds

$$\exp(\mathbf{D}t)\mathbf{g}(\mathbf{y}) \equiv \mathbf{g}(\exp(\mathbf{D}t)\mathbf{y})$$

The Poincaré–Dulac theorem [3] states that any system may be reduced to Poincaré normal form. Consequently, the linear non-autonomous substitution $\mathbf{z} = \exp(\mathbf{D}t)\mathbf{y}$ reduces (1.4) to the system

$$\mathbf{z}^{\cdot} = \mathbf{g}(\mathbf{z}) \tag{1.5}$$

whose linear part is nilpotent.

Strictly speaking, the relation between non-exponential asymptotic solutions of system (1.5) and the solutions of the initial system (1.1) needs additional justification, since the normalizing transformation may diverge. Without going into details, we merely note that a proof may be based on the abstract implicit function theorem, as was one in [4].

Thus, our main goal is to look for asymptotic solutions of non-exponential type for systems of differential equations (1.1) with nilpotent linear part. We shall solve an even more general problem. Let the vector field $\mathbf{f}(\mathbf{x})$ be such that its components $f^1, \dots, f^n$ may be expressed as formal (finite or infinite) power series

$$f^j = \sum_{i_1,\dots,i_n} f^j_{i_1,\dots,i_n} (x^1)^{i_1} \dots (x^n)^{i_n} \tag{1.6}$$

where the indices $i_1, \dots, i_n$ are non-negative integers. At this stage we shall not even assume that $\mathbf{x} = \mathbf{0}$ is an equilibrium position of the initial system. We shall find sufficient conditions for such systems to have particular solutions whose components have a generalized power-type asymptotic behaviour either as $t \to \pm 0$ or as $t \to \pm\infty$ (in the first case, since the system is autonomous, any finite number may be substituted for zero). In this situation, it may not be sufficient to require the functions on the right of (1.1) to be smooth in only a small neighbourhood of $\mathbf{x} = \mathbf{0}$. We shall therefore assume throughout, without further mention, that these functions are all infinitely differential in that region of space where the required solutions are hypothetically defined.

To carry out the above task we shall use the three-step scenario of Lyapunov's First Method as described above.

2. ELECTION OF A MODEL TRUNCATED SYSTEM

At this step we shall single out from system (1.1) some simpler subsystem which in a certain sense reproduces the former's properties. The "simplicity" of the truncated or, as it is also called, model system means that it possesses certain symmetry properties that enable one to find particular solutions in explicit form.

If the matrix of the linear approximation were the zero matrix, the most natural way to "truncate" the system would be to retain the terms of the first non-trivial order, i.e. to pick out a homogeneous subsystem. Unfortunately, it is far from certain that this approach will produce desirable results. In the "strongly non-linear" theory one also uses "quasi-homogeneous" systems as truncated systems. These systems possess a symmetry group of quasi-homogeneous dilatations, which are singled out from the full system by using Newton polygons of the indices of the expansions (1.6) [5, 6].

Let us generalize the concepts and definitions of [5, 6]. Consider a linear system of differential equations of Fuchsian type

$$\mu \frac{d\mathbf{x}}{d\mu} = \mathbf{G}\mathbf{x}, \quad \mu \frac{dt}{d\mu} = -t \tag{2.1}$$

where μ is a real parameter and $\mathbf{G}$ is a real (not necessarily diagonal) matrix whose flow will be denoted as

$$\mathbf{x} \mapsto \mu^{\mathbf{G}}\mathbf{x}, \quad t \mapsto \mu^{-1}t \tag{2.2}$$

Definition 2. A vector field $\mathbf{f}(\mathbf{x}) = \mathbf{f}_q(\mathbf{x})$ is said to be quasi-homogeneous if the corresponding system of differential equations is invariant under the action of the phase flow (2.2) of the Fuchsian system (2.2).

Below we shall assign a quasi-homogeneity subscript q to the right-hand sides of quasi-homogeneous systems. In a few cases we shall use this term for a certain generalization of the "degree" of quasi-homogeneity.

If $\mathbf{G}$ is a diagonal matrix and its elements are positive rational numbers, Definition 2 is essentially the same as the standard definition of a quasi-homogeneous system [5, 6]. The need to consider systems (2.1) with non-diagonal matrices $\mathbf{G}$ will be demonstrated later by specific examples.

We will examine the question of the type of systems for which quasi-homogeneous systems

$$\mathbf{x}^{\cdot} = \mathbf{f}_q(\mathbf{x}) \tag{2.3}$$

may be used as model systems.

Definition 3. A system of equations (1.1) is said to be semiquasi-homogeneous if the flow (2.2) transforms its right-hand side to

$$\mathbf{x}^{\cdot} = \mathbf{f}_q(\mathbf{x}) + \mathbf{f}^{*}(\mathbf{x}, \mu) \tag{2.4}$$

where $\mathbf{f}_q(\mathbf{x})$ is some quasi-homogeneous vector field and $\mathbf{f}^{*}(\mathbf{x}, \mu)$ is a formal power series in μ^{β}, $\beta \in \mathbb{R}\backslash\{0\}$ without a free term. If $\beta > 0$, system (1.1) will be called positively semiquasi-homogeneous; it will be called negatively semiquasi-homogeneous if β is negative.

Anticipating, we note that if a system is positively semiquasi-homogeneous, one should look for particular solutions that possess generalized power-type asymptotic behaviour as $t \to \pm\infty$; if it is negatively semiquasi-homogeneous, one should investigate the asymptotic behaviour of the solutions as $t \to \pm 0$. Negatively semiquasi-homogeneous systems generally have only a finite number of terms in the expansion (1.6), that is, they are polynomial systems.

Let us consider two simple examples.

Example 1. The system of equations

$$x^{\cdot} = y, \quad y^{\cdot} = ax^2$$

is quasi-homogeneous relative to the flow (2.2) of the Fuchsian system (2.1) with matrix $\mathbf{G} = \operatorname{diag}(2, 3)$. On the other hand, any system of the form

$$x^{\cdot} = y + X(x, y), \quad y^{\cdot} = ax^2 + Y(x, y)$$

where $X(0, 0) = Y(0, 0) = 0$, X, Y contain no linear terms and in addition Y contains no quadratic terms in x, is semiquasi-homogeneous.

Example 2. The system of equations

$$x^{\cdot} = (x^2 + y^2)(ax - by), \quad y^{\cdot} = (x^2 + y^2)(ay + bx)$$

is of course homogeneous of degree 3. On the other hand, it remains invariant under the action of phase flows of the following family of Fuchsian systems of equations

$$\mu dx/d\mu = {}^1/_2 x + \delta y, \quad \mu dy/d\mu = {}^1/_2 y - \delta x, \quad \mu dt/d\mu = -t$$

Hence it is quasi-homogeneous in the sense of definition 2. Furthermore, any system of the form

$$x^{\cdot} = (\rho + f(\rho))(ax - by), \quad y^{\cdot} = (\rho + f(\rho))(ay + bx)$$

where $\rho = x^2 + y^2$ and $f(\rho) = o(\rho)$ as $\rho \to 0$, is semi-quasi homogeneous.

Thus, in order to find all non-trivial truncations of the initial system (1.1), one must determine a family of matrices $\mathbf{G}$ such that the system takes the form (2.4) under the action of the flow (2.2).

3. PARTICULAR SOLUTIONS OF THE MODEL SYSTEMS

Since system (2.3) is quasi-homogeneous, under fairly general conditions it has a particular solution that is a "quasi-homogeneous ray"

$$\mathbf{x}^{\gamma}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{x}_0^{\gamma} \tag{3.1}$$

where $\gamma = \pm 1$ and $\mathbf{x}_0^{\gamma}$ is a constant real vector. Later we shall make this notion of "fairly general conditions" more specific.

If (2.3) has a particular solution of the form (3.1), the vector $\mathbf{x}^{\gamma}{}_0$ must satisfy the following algebraic system of equations

$$-\gamma\mathbf{G}\mathbf{x}_0^{\gamma} = \mathbf{f}_q(\mathbf{x}_0^{\gamma}) \tag{3.2}$$

The vector $\mathbf{x}_0^{\gamma}$ will be called an eigenvector of the quasi-homogeneous vector field $\mathbf{f}_q(\mathbf{x})$ generated by the quasi-homogeneous structure of $\mathbf{G}$. The particular solution (3.1) of the truncated system (2.3) corresponds to an exponential particular solution of the linear system, whose existence follows from the existence of a particular solution of the full system with exponential asymptotic behaviour.

In the Lyapunov case, the determination of eigenvectors reduces to a standard problem of linear algebra. In this non-linear case, however, the search for eigenvectors may be far from easy. Nonetheless, in some cases their existence follows from fairly simple geometrical considerations.

Assume that $\mathbf{G}$ is non-singular. Definition 2 immediately implies the identity

$$\mathbf{f}_q(\mu^{\mathbf{G}}\mathbf{x}) = \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_q(\mathbf{x}) \tag{3.3}$$

We shall seek $\mathbf{x}_0^{\gamma}$ in the form

$$\mathbf{x}_0^{\gamma} = \mu^{\mathbf{G}}\mathbf{e}^{\gamma}$$

where μ is a positive number and $\mathbf{e}^{\gamma} \in \mathbb{R}^n$ $\|\mathbf{e}^{\gamma}\| = 1$ is a certain unit vector.

Equation (3.2) then becomes

$$\mathbf{G}^{-1}\mathbf{f}_q(\mathbf{e}^{\gamma}) = -\gamma\mu^{-1}\mathbf{e}^{\gamma} \tag{3.4}$$

Let $\mathbf{x} = \mathbf{0}$ be the unique critical point of the quasi-homogeneous vector field $\mathbf{f}_q(\mathbf{x})$. Consider the Gaussian mapping

$$\Gamma(\mathbf{p}) = \frac{\mathbf{G}^{-1}\mathbf{f}_q(\mathbf{p})}{\left\| \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{p}) \right\|}, \quad \mathbf{p} \in S^{n-1} \tag{3.5}$$

of the sphere S^{n-1} into itself.

The following lemma is a quasi-homogeneous analogue of propositions formulated in [4, 7].

Lemma 1. Let $\mathbf{x} = \mathbf{0}$ be the unique critical point of a quasi-homogeneous vector field $\mathbf{f}_q$: $\mathbb{R}^n \to \mathbb{R}^n$.

1. If the degree of the Gaussian mapping Γ is even, then $\mathbf{f}_q$ has eigenvectors with both positive and negative eigenvalues γ.

2. If the phase space is of odd dimension n, then $\mathbf{f}_q$ has at least one eigenvector with either positive or negative eigenvalue.

Proof. The first statement is based on a well-known topological result [8], according to which a Gaussian mapping of the sphere into itself of degree other than $(-1)^d$, where d is the dimension of the sphere, always has a fixed point. Applying this result to the mapping $-\Gamma$, we see that Γ also has an antipodal point. Consequently, unit vectors $\mathbf{e}^+$ and $\mathbf{e}^-$ exist for which equality (3.4) holds with

$$\mu = \left\| \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{e}^\gamma) \right\|^{-1}$$

In order to prove part 2, it will suffice to consider the tangent vector field

$$\mathbf{v}(\mathbf{p}) = \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{p}) - \langle \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{p}), \mathbf{p} \rangle \mathbf{p}, \quad \mathbf{p} \in S^{n-1}$$

on a sphere S^{n-1} of even dimension (throughout, the symbol $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$), on which no smooth vector field exists without zeros [8]. Hence a vector $\mathbf{e}^\gamma \in \mathbb{R}^n$ exists such that (3.4) holds with

$$\mu = \left| \langle \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{e}^\gamma), \mathbf{e}^\gamma \rangle \right|^{-1}, \quad \gamma = -\operatorname{sign}\langle \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{e}^\gamma), \mathbf{e}^\gamma \rangle$$

As the vector $\mathbf{G}^{-1}\mathbf{f}_q(\mathbf{e}^\gamma)$ is parallel to $\mathbf{e}^\gamma$ and $\mathbf{x} = \mathbf{0}$ is the unique critical point of the vector field $\mathbf{f}_q$, it follows that $\mu^{-1} \neq 0$.

We shall now discuss the question of why it is necessary to consider semiquasi-homogeneous systems whose quasi-homogeneous structure is defined by a non-diagonal matrix $\mathbf{G}$.

Suppose that system (1.1) was obtained by reducing an initial system to a central manifold, converting it to Poincaré form, and dropping those of the linear terms that correspond to the diagonalizable component $\mathbf{D}$ of matrix $\mathbf{A}$, all of whose eigenvalues lie on the imaginary axis. Suppose that system (1.1) is semiquasi-homogeneous relative to the quasi-homogeneous structure generated by some matrix $\mathbf{G}$.

Lemma 2. Let the matrices $\mathbf{D}$ and $\mathbf{G}$ commute, i.e. $\mathbf{DG} = \mathbf{GD}$. Then, under the above assumptions, system (1.1) is semiquasi-homogeneous relative to the family of quasi-homogeneous structures generated by the family of matrices $\mathbf{G}_\delta = \mathbf{G} + \delta\mathbf{D}$, where δ is the parameter of the family, with the same truncation.

Proof. It is obvious that the right-hand sides of (1.1) admit of formal expansions as series of "quasi-homogeneous forms"

$$\mathbf{f}(\mathbf{x}) = \sum_{m=0} \mathbf{f}_{q+\chi m}(\mathbf{x})$$

The following identities, which are generalizations of (3.3), hold

$$\mathbf{f}_{q+\chi m}(\mu^{\mathbf{G}}\mathbf{x}) = \mu^{\mathbf{G}+(1+\beta m)\mathbf{E}}\mathbf{f}_{q+\chi m}(\mathbf{x}), \quad \chi = \operatorname{sign}\beta \tag{3.6}$$

Using the definition of Poincaré normal form and the commuting property of $\mathbf{D}$ and $\mathbf{G}$, and putting $\mu = e^{t/\delta}$, we obtain

$$\mathbf{f}(\mu^{\mathbf{G}_\delta}\mathbf{x}) = \mathbf{f}(\exp(\mathbf{D}t)\mu^{\mathbf{G}}\mathbf{x}) = \exp(\mathbf{D}t)\mathbf{f}(\mu^{\mathbf{G}}\mathbf{x}) = \mu^{\delta\mathbf{D}+\mathbf{G}+\mathbf{E}} \sum_{m=0} \mu^{\beta m}\mathbf{f}_{q+\chi m}(\mathbf{x})$$

as required.

Suppose that, for simplicity, the truncated model system (2.30) is homogeneous of degree $q \geqslant 2$. On the other hand, this system is quasi-homogeneous relative to the flow (2.2) of a Fuchsian system (2.1) with matrix $\mathbf{G}_\delta$. The investigation of this structure is dictated by the following considerations. The truncated homogeneous system may not have real particular solutions in the form of a straight ray

$$\mathbf{x}^\gamma(t) = (\gamma t)^{-\beta}\mathbf{x}_0^\gamma$$

However, for some non-zero δ it may have real solutions in the form of a "twisted" ray

$$\mathbf{x}^\gamma(t) = (\gamma t)^{-\beta}(\gamma t)^{-\delta\mathbf{D}}\mathbf{x}_0^\gamma$$

This is the situation, for example, in Hamiltonian systems with resonances of even degrees [9].

4. CONSTRUCTION OF PARTICULAR SOLUTIONS OF THE FULL SYSTEM

We shall show that, as in the quasi-linear case, the scenario of Lyapunov's First Method enables us to complete particular solutions (3.1) of the truncated system to solutions of the full system, as certain series.

We formally express the solution of Eqs (1.1) as a series

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k(\ln(\gamma t))(\gamma t)^{-k\beta} \tag{4.1}$$

where the coefficients $\mathbf{x}_k$ are polynomial vector-functions.

To prove that a formal particular solutions of this kind exists, we change the dependent and independent variables in (1.1), using identity (3.6)

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(\gamma t), \quad s = (\gamma t)^{-\beta}$$

as a result of which the initial system (1.1) becomes

$$-\gamma\beta s\mathbf{y}' = \gamma\mathbf{G}\mathbf{y} + \sum_{m=0} s^m\mathbf{f}_{q+\chi m}(\mathbf{y}) \tag{4.2}$$

where the prime denotes differentiation with respect to the new independent variable s.

The formal solution of system (4.2) then becomes a series

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{x}_k\left(-\tfrac{1}{\beta}\ln s\right)s^k \tag{4.3}$$

We now substitute (4.3) into (4.2) and equate the coefficients of s^k. Assuming that the zeroth coefficient $\mathbf{x}_0$ is a constant and putting $k = 0$, we obtain precisely Eq. (3.2). Hence the fact that the model system has a particular solution of type (3.1) guarantees that one can determine the zeroth coefficient in expansion (4.1). For higher values of k one has the following systems of equations

$$\gamma\frac{d\mathbf{x}_k}{d\tau} - \mathbf{K}_k\mathbf{x}_k = \mathbf{\Phi}_k(\mathbf{x}_0,\dots,\mathbf{x}_{k-1}) \tag{4.4}$$

where $\mathbf{\Phi}_k$ are certain polynomial vector-valued functions of their arguments, $\tau = -\beta^{-1}\ln s = \ln(\gamma t)$, the matrix $\mathbf{K}_k = k\gamma\beta\mathbf{E} + \mathbf{K}$, and

$$\mathbf{K} = \gamma\mathbf{G} + d\mathbf{f}_q(\mathbf{x}_0^\gamma) \tag{4.5}$$

is the so-called Kovalevskaya matrix [10].

If one assumes that all the coefficients up to the kth have been found and are polynomials in τ, then $\mathbf{\Phi}_k$ are certain known polynomials in τ. The resulting system (4.4) may be regarded as a system of ordinary differential equations with constant coefficients and a polynomial right-hand side which, as is well known, always has a particular solution $\mathbf{x}_k(\tau)$ which is a polynomial of degree $N_k + S_k$, where N_k is the degree of $\mathbf{\Phi}_k$ as a polynomial in τ and S_k is the multiplicity of zero in the sequence of eigenvalues of $\mathbf{K}_k$. Thus, all the coefficients of the series (4.3) may be determined by induction. The formal construction of a partial asymptotic solution of (1.1) in the form (4.1) is thus complete.

Note that in the positively semiquasi-homogeneous case ($\beta > 0$) the logarithms in expansions (4.1) cannot generally be eliminated. Using the technique of [10], it can be shown that $-\gamma$ is always an eigenvalue of the Kovalevskaya matrix (4.5). In specific applications $\beta = 1/(q-1)$, where $q \geqslant 2$ is the generalized degree of quasi-homogeneity; hence the matrix $\mathbf{K}_{q-1}$ is always singular, so that logarithms will always appear in the general case.

The next step should be to prove the existence of an infinitely differentiable particular solution of system (1.1) for which (4.1) is an asymptotic expansion. In various situations this can be done either by relying on results of [11] or can be proved directly by the technique of [4].

We have thus proved the following theorem.

Theorem 1. Let system (1.1) be semiquasi-homogeneous and suppose a vector $\mathbf{x}_0^\gamma \in \mathbb{R}^n$, $\mathbf{x}_0^\gamma \neq \mathbf{0}$ and a number $\gamma = \pm 1$ exist such that (3.2) holds. Then system (1.1) has a particular solution, the leading term of whose asymptotic expansion is $(\gamma t)^{-\mathbf{G}}\mathbf{x}_0^\gamma$ as $t^\chi \to \gamma \times \infty$, where $\chi = \operatorname{sign} \beta$ is the "sign of quasi-homogeneity".

We will now make a few characteristic observations that demonstrate the similarity of the above method to the classical version of Lyapunov's First Method. Applying the time change $\tau = \ln(\gamma t)$ in (4.1), one actually obtains exponential series with coefficients that are polynomials in τ, that is, those used in Lyapunov's First Method. Logarithmic time is thus natural for strongly non-linear systems.

Using Lyapunov's First Method one can establish the existence of not only one particular solution but of a whole family of solutions that exponentially approach the position of equilibrium. We will formulate the analogous result for strongly non-linear systems.

Theorem 2. Suppose that all the conditions of Theorem 1 are satisfied and that p eigenvalues of the Kovalevskaya matrix $\mathbf{K}$ have real parts with the same sign as that of the quantity $-\gamma\chi$; the real parts of the other eigenvalues either vanish or have the opposite sign. Then system (1.1) has a p-parameter family of particular solutions of the form

$$\mathbf{x}(\mathbf{c}, t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^\gamma + o(1)) \quad \text{as} \quad t^\chi \to \gamma \times \infty$$

where $\mathbf{c} \in \mathbb{R}^p[\mathbf{c}]$ is a parameter vector.

The proof is based on the reduction of system (1.1) to quasi-linear form. Indeed, after the substitution

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^\gamma + \mathbf{u}(\gamma t)), \quad \tau = \ln(\gamma t)$$

system (3.2) may be rewritten as

$$\gamma \frac{d\mathbf{u}}{d\tau} = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{u}, \tau) \tag{4.6}$$

The vector functions $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ have the following properties: $\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}$, $d\boldsymbol{\psi}(\mathbf{0}) = \mathbf{0}$, and the function $\boldsymbol{\psi}$ may be expressed as a power series in $e^{-\beta\tau}$ without free term, so that $\boldsymbol{\psi}(\mathbf{u}, \chi \times \infty) \equiv \mathbf{0}$.

Consequently, in order to prove the above statement one can use the technique of [12], i.e. reduce (4.6) to a system of integral equations and use the principle of contractive mappings.

The actual form of the families of solutions mentioned in Theorem 2 is rather complicated, especially if one is interested in real solutions only; we shall therefore not present it here.

5. EXAMPLE: INVERSION OF THE LAGRANGE–DIRICHLET THEOREM

We will show how the method described above may be used to prove one of the known inversions of the Lagrange–Dirichlet stability theorem [13, 14].

Consider the motion of a mechanical system described by a Hamiltonian system of equations

$$\mathbf{p}^{\cdot} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \mathbf{q}^{\cdot} = \frac{\partial H}{\partial \mathbf{p}}, \quad (\mathbf{p},\mathbf{q}) \in \mathbb{R}^n \times \mathbb{R}^n \tag{5.1}$$

where the Hamiltonian is the sum of the kinetic and potential energies, $H = T + U$

$$T = \tfrac{1}{2}\langle \mathbf{K}(\mathbf{q})\mathbf{p}, \mathbf{p}\rangle, \quad U = U(\mathbf{q})$$

Suppose that the potential energy and the components of the kinetic energy matrix are infinitely differentiable functions in the neighbourhood of the origin $\mathbf{q} = \mathbf{0}$, where the latter is a position of equilibrium ($dU(\mathbf{0}) = 0$). We may also assume without loss of generality that $U(\mathbf{0}) = 0$ and $\mathbf{K}(\mathbf{0}) = \mathbf{E}$.

Suppose that the Maclaurin expansion of the potential energy begins with terms of order $q + 1$, $q \geqslant 2$. Then the system of equations (5.1) is semiquasi-homogeneous relative to the structure defined by the matrix

$$\mathbf{G} = \mathrm{diag}((q+1)\beta, \ldots, (q+1)\beta, 2\beta, \ldots, 2\beta), \quad \beta = 1/(q-1)$$

The truncated system is

$$\mathbf{p}^{\cdot} = -\frac{\partial U_{q+1}}{\partial \mathbf{q}}(\mathbf{q}), \quad \mathbf{q}^{\cdot} = \mathbf{p} \tag{5.2}$$

If the first non-trivial form U_{q+1} in the expansion of the potential energy does not have a minimum, then the model system (5.2) has a particular solution that is a quasi-homogeneous ray

$$\mathbf{p}^{+}(t) = -2\beta \mathbf{c} t^{-(q+1)\beta}, \quad \mathbf{q}^{+}(t) = \mathbf{c} t^{-2\beta}, \quad \mathbf{c} \in \mathbb{R}^n$$

where the vector $\mathbf{c}$ is parallel to a unit vector which minimizes the homogeneous form U_{q+1} on the unit sphere about zero.

Consequently, the full system (5.1) has a particular solution $(\mathbf{p}(t), \mathbf{q}(t)) \to (\mathbf{0}, \mathbf{0})$ as $t \to +\infty$, whence it follows, since (5.1) is invertible, that the equilibrium position is unstable.

6. EXAMPLE: EXTENSION OF THE LYAPUNOV CRITERION

We shall now derive sufficient conditions for instability of the position of equilibrium of system of differential equations whose linear part is a Jordan block of dimension greater than two with zero diagonal. An exhaustive criterion for instability in the two-dimensional case was found by Lyapunov [1].

We will write the system of equations being investigated as

$$x_i^{\cdot} = x_{i+1} + \ldots, i = 1, \ldots, n-1; \quad x_n^{\cdot} = a x_1^2 + \ldots \tag{6.1}$$

where the dots stand for the non-linear terms, from which the monomial ax_1^2 has been singled out. It is easy to see that the system is quasi-homogeneous relative to the structure defined by the diagonal matrix

$$\mathbf{G} = \mathrm{diag}(n, n+1, \ldots, 2n-1)$$

It can be shown that system (6.1) is positively semiquasi-homogeneous ($\chi = +1$). If $a \neq 0$, the quasi-homogeneous truncation will have a particular asymptotic solution

$$\mathbf{x}^{+}(t) = t^{-\mathbf{G}} \mathbf{x}_0^{+}, \quad \text{or} \quad x_i^{+}(t) = x_{0_i}^{+} / t^{n+i+1}, \quad i = 1, \ldots, n$$

where

$$x_{0_i}^+ = (-1)^{n+i-1}\frac{(2n-1)!(n+i-2)!}{a((n-1)!)^2},\quad i=1,\ldots,n$$

whence it follows that the full system has a particular asymptotic solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to +\infty$.

Similarly, one can prove the existence of a particular asymptotic solution of (6.1) such that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$, implying instability.

Suppose that there is an additional degeneracy in system (6.1) ($a = 0$). We will confine our attention to the case $n \geqslant 3$.

Rewrite system (6.1) as follows:

$$\dot{x}_i = x_{i+1}+\ldots, i=1,\ldots,n-2;\quad \dot{x}_{n-1} = x_n + bx_1^2+\ldots,\quad \dot{x}_n = 2cx_1x_2+\ldots \tag{6.2}$$

Again, the dots stand for the non-linear terms, from which the monomial bx_1^2 has been singled out in the penultimate equation and the monomial $-2cx_1x_2$ in the last equation. The system thus obtained is quasi-homogeneous with the diagonal matrix $\mathbf{G} = \operatorname{diag}(n-1, n, \ldots, 2n-2)$. If $b + c \neq 0$, then system (6.2) has a particular asymptotic solution of the form

$$\mathbf{x}^+(t) = t^{-\mathbf{G}}\mathbf{x}_0^+,\quad \text{or}\quad x_i^+(t) = x_{0_i}^+ / t^{n+i-2},\quad i=1,\ldots,n$$

$$x_{0_i}^+ = (-1)^{n+i}\frac{(2n-3)!(n+i-3)!}{(b+c)(n-2)!^2},\quad i=1,\ldots,n-1,\quad x_{0_n}^+ = c\left(\frac{(2n-3)!}{(b+c)(n-2)!}\right)^2$$

It can be shown that the full system is positively semiquasi-homogeneous, and so system (6.2) has an asymptotic solution such that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to +\infty$. One can also prove the existence of a solution that tends to $\mathbf{x} = \mathbf{0}$ as $t \to -\infty$, which guarantees instability.

Generally speaking, in order to prove the existence of asymptotic solutions of system (6.1) as $t \to +\infty$ and $t \to -\infty$, it should be sufficient to use Lemma 2. By virtue of certain symmetry properties of the right-hand side of the system, any non-critical value of the Gaussian mapping (3.5) has double pre-images, and therefore the degree of the mapping is even.

7. EXAMPLE: ASYMPTOTIC TRAJECTORIES OF GENERAL SYSTEMS OF DIFFERENTIAL EQUATIONS WITH 1:1 FREQUENCY RESONANCE AND NON-SIMPLE ELEMENTARY DIVISORS

We will consider this problem to illustrate Lemma 2. The stability of such systems was studied in [6]. The model system is [6]

$$\begin{aligned} &\dot{x}_1 = x_2, && \dot{y}_1 = y_2 \\ &\dot{x}_2 = (ax_1 - by_1)(x_1^2+y_1^2), && \dot{y}_2 = (bx_1+ay_1)(x_1^2+y_1^2) \end{aligned} \tag{7.1}$$

It is obvious that this system is quasi-homogeneous with the matrix $\mathbf{G} = \operatorname{diag}(1, 2, 2, 2)$, which commutes with the diagonalizable part, $\mathbf{D}$, say, of the matrix of the linear approximation. Hence the full system is semiquasi-homogeneous relative to the structures generated by the family of matrices $\mathbf{G}_\delta = \mathbf{G} + \delta\mathbf{D}$. It can be verified that system (7.1) does not have real particular solutions that are quasi-homogeneous rays.

We will seek particular solutions of the system that are "twisted" rays. Since the equations of the model system are invertible, the conditions for such solutions to exist are the same for $\gamma = +1$ as for $\gamma = -1$; the index γ will therefore be omitted.

Thus

$$x_s = t^{-s}\left(x_{0_s}\cos\varphi - y_{0_s}\sin\varphi\right),\quad y_s = t^{-s}\left(x_{0_s}\sin\varphi + y_{0_s}\cos\varphi\right)$$
$$s = 1, 2;\qquad \varphi = \delta\ln t$$

Then the system of algebraic equations (3.2) has a one-parameter family of solutions

$$x_{0_1} = \rho\cos\theta,\quad y_{0_1} = \rho\sin\theta;$$

$$x_{0_2} = \rho(\delta\sin\theta - \cos\theta), \quad y_{0_2} = -\rho(\delta\cos\theta + \sin\theta)$$

where θ is the parameter and δ, $\rho > 0$ satisfy the system of equations

$$a\rho^2 + \delta^2 - 2 = 0, \quad 3\delta - b\rho^2 = 0$$

In complex form the conditions for this system to be solvable are $c \neq -|c|$ $(c = a + ib)$, i.e. precisely the instability conditions found in [6].

If these conditions are satisfied, the initial full system of differential equations has particular solutions that tend to the equilibrium position both as $t \to +\infty$ and as $t \to -\infty$.

8. EXAMPLE: COLLISION TRAJECTORIES IN HILL'S PROBLEM

Up to this point we have considered only positively semiquasi-homogeneous systems. The equations of Hill's problem are interesting because they may be treated as a negatively quasi-homogeneous system.

Consider a Hamiltonian system of equations with Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + p_x y - p_y x - x^2 + \tfrac{1}{2}y^2 - (x^2 + y^2)^{-1/2} \tag{8.1}$$

These equations describe the plane motion of a satellite of low mass, such as the Moon, in the field of attraction of two bodies, one of low mass compared with that of the other, such as the Earth and the Sun. A detailed formulation of the problem may be found in [15]. The equations with Hamiltonian (8.1) also describe, to within a certain approximation, the relative motion of two satellites of a massive attracting body along nearby orbits [16].

Introducing a new auxiliary variable $s = (x^2 + y^2)^{-1/2}$, we obtain a polynomial system of equations of order five

$$\begin{aligned} p_x^{\cdot} &= p_y + 2x - xs^3, \quad x^{\cdot} = p_x + y \\ p_y^{\cdot} &= -p_x - y - ys^3, \quad y^{\cdot} = p_y - x \\ s^{\cdot} &= -xp_x s^3 - yp_y s^3 \end{aligned} \tag{8.2}$$

Introducing a quasi-homogeneous scale for the phase variables, $p_x \mapsto \mu^{g_{p_x}} p_x$, $x \mapsto \mu^{g_x} x$, $p_y \mapsto \mu^{g_{p_y}} p_y$, $y \mapsto \mu^{g_y} y$, $s \mapsto \mu^{g_s} s$, $t \mapsto \mu^{-1} t$, let us determine the possible truncations.

If we take

$$g_{p_x} = -\tfrac{2}{3}, \quad g_x = -\tfrac{5}{3}, \quad g_{p_y} = \tfrac{1}{3}, \quad g_y = -\tfrac{2}{3}, \quad g_s = \tfrac{2}{3}$$

the system of equations (8.2) takes a form from which it follows that it is negatively semiquasi-homogeneous relative to the "scale" introduced above. Setting $\mu = \infty$, we obtain a truncated quasi-homogeneous system which has a particular solution

$$p_x = p_{x_0} t^{2/3}, \quad x = x_0 t^{5/3}, \quad p_y = p_{y_0} t^{-1/3}, \quad y = y_0 t^{2/3}, \quad s = s_0 t^{-2/3} \tag{8.3}$$

where

$$p_{x_0} = \pm\tfrac{2}{3} s_0^{-1}, \quad x_0 = \pm s_0^{-1}, \quad p_{y_0} = \pm\tfrac{2}{3} s_0^{-1}, \quad y_0 = \pm s_0^{-1}, \quad s_0 = (\tfrac{2}{9})^{1/3} \tag{8.4}$$

In accordance with Theorem 1, the system of equations with Hamiltonian (8.1) has a particular solution with the asymptotic expansion (summation from $k = 0$ to $k = \infty$)

$$\begin{aligned} p_x(t) &= t^{2/3} \sum p_{x_k} t^{k/3}, \quad x(t) = t^{5/3} \sum x_k t^{k/3} \\ p_y(t) &= t^{-1/3} \sum p_{y_k} t^{k/3}, \quad y(t) = t^{2/3} \sum y_k t^{k/3} \end{aligned} \tag{8.5}$$

But if we take

$$g_{p_x} = 1/3,\quad g_x = -2/3,\quad g_{p_y} = -2/3,\quad g_y = -5/3,\quad g_s = 2/3$$

then system (8.2) with $\mu = \infty$ becomes a truncated system that has a particular solution of type (8.3) with x and y interchanged, where p_{x0}, x_0, s_0 are the same as in (8.4) but p_{y0}, y_0 are given the opposite sign.

Consequently, the initial system of equations has a particular solution with an asymptotic expansion analogous to (8.5), the first terms of which were described above.

One prerequisite for using the algorithm of Section 4 to construct formal asymptotic series (4.1), in its general form, is that the coefficients of expansion (8.5) are polynomials in $\ln t$. Simple, though tedious, calculations show that the coefficients p_{xk}, x_k, p_{yk}, y_k are constants.

The trajectories of Hill's problem corresponding to particular solutions with asymptotic expansions (8.5) are called collision trajectories. Along such trajectories, the smaller of the attracting bodies and the satellite (Earth and Moon) will collide within a finite time (relative to the selected origin as $t \to 0$). The algorithm of Section 4 yields a recurrent procedure to determine the coefficients of the expansions and, consequently, a constructive technique to obtain collision trajectories in "real time". Previous treatments of collisions in Hill's problem used regularizing changes of variables, as introduced by Birkhoff [17], which lead to Hamiltonian systems which depend on the energy constant as a parameter. In that context the true motions may occur only at the zero energy level of the new system, which is, of course, not very convenient for investigation.

9. BIBLIOGRAPHICAL COMMENTS

Strange though it may seem, the earliest published work on particular solutions of systems of differential equations with non-exponential asymptotic expansions appeared long before Lyapunov's *The General Problem of the Stability of Motion*, which constructs a theory of solutions with exponential asymptotic expansions. We are referring primarily to the work of Briot and Bouquet [18]. Using the technique of [18], Kamenkov proved a stability theorem [19] based on an idea not unlike that of our Theorem 1. Kamenkov's result differs from ours in the following respects: first, all his truncated systems are homogeneous; second, he did not construct solutions of a system of type (1.1) as series (4.1) in the time variable, but certain invariant curves that approach the position of equilibrium, as series similar to (4.1) but in a certain auxiliary variable x_1; third, as Kamenkov was interested only in stability problems, his technique is not suitable for constructing solutions that admit ordinary power-type asymptotic series as $t \to 0$. Unfortunately, Kamenkov's theorem remained largely unknown, and it was therefore proved anew, with minimal changes, in [20], without reference to Kamenkov [19]. Another result of [20] was an analogue of Theorem 2 for homogeneous truncations. One of the first publications in which it was pointed out that homogeneous truncations do not always determine the behaviour pattern of the trajectories in the neighbourhood of a singular point was [21]. In that paper it was proposed that quasi-homogeneous systems could be employed as truncated systems, with Newton polygons used to single out such systems. A method for the constructive determination of solutions of the truncated systems as quasi-homogeneous rays was also indicated in [21]. However, the problem of completing these solutions to solutions of the full system was not discussed either in [21] or in later research in that direction. The approach based on defining truncations using flows of Fuchsian systems is apparently new.

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